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Introduction

Let X be a geometrically connected smooth projective curve over a field K. Let $Y := X \times X \times X$ be the triple product of X over K and let

$$e = \sum_i a_i p_i \quad (a_i \in \mathbb{Q})$$

be a $\mathbb{Q}$-divisor on X of degree 1. For such an e, Gross and Schoen defined in [5] a modified diagonal cycle Δ_e on Y with respect to the base e as follows: put

$$\begin{aligned}
\Delta_{123} &:= \{(x,x,x) \mid x \in X\} \\
\Delta_{12} &:= \sum_i a_i \{(x,x,p_i) \mid x \in X\} \\
\Delta_{23} &:= \sum_i a_i \{(p_i,x,x) \mid x \in X\} \\
\Delta_{31} &:= \sum_i a_i \{(x,p_i,x) \mid x \in X\} \\
\Delta_{1} &:= \sum_{i,j} a_i a_j \{(x,p_i,p_j) \mid x \in X\} \\
\Delta_{2} &:= \sum_{i,j} a_i a_j \{(p_i,x,p_j) \mid x \in X\} \\
\Delta_{3} &:= \sum_{i,j} a_i a_j \{(p_i,p_j,x) \mid x \in X\},
\end{aligned}$$

and define Δ_e by

$$\Delta_e := \Delta_{123} - \Delta_{12} - \Delta_{23} - \Delta_{31} + \Delta_1 + \Delta_2 + \Delta_3.$$

They have shown that Δ_e is homologically trivial and it, as an element of the $\mathbb{Q}$-Chow group of Y, depends only on the $\mathbb{Q}$-linear equivalence class of e. When $e = \xi$ has an additional property that $(2g-2)\xi - K_X$ is $\mathbb{Q}$-linearly equivalent to 0, we call Δ_ξ the *canonical Gross-Schoen cycle.*

Now let us suppose that K is a global field, that is, a finite extension of $\mathbb{Q}$ or a function field of a smooth projective curve over an algebraically closed base field k. Gross and Schoen defined in [5] the height $\langle \Delta_e, \Delta_e \rangle$ of Δ_e via construction of a “good” model, and Zhang studied it in detail from the view point of intersection of admissible line bundles in [13].

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Following Zhang's method, we will discuss in this paper its positivity for some kinds of curves X and give some applications when K is a function field. In the sequel, we denote the canonical Gross-Schoen cycle Δ_ξ by Δ for simplicity.

The calculation of the height is quite important for many aspects of arithmetic problems for varieties. From the viewpoint of the Beilinson-Bloch conjecture for Y for example, we are interested in the rank of the subgroup of the Chow group consisting of the homologically trivial cycles, and hence we like to construct many non-trivial homologically trivial cycles. Therefore we have a natural question whether the Gross-Schoen cycle is rationally trivial or not. Since the height pairing vanishes for rationally trivial cycles, it is then natural to ask whether the height of the Gross-Schoen cycle is 0 or not. Furthermore, since the height $\langle \Delta_e, \Delta_e \rangle$ takes its minimum when $\Delta_e = \Delta$, we find that the positivity of $\langle \Delta, \Delta \rangle$ is an important problem. Note that it is known to be non-negative when K is a function field of characteristic 0, but it is not proved when K is a function field of positive characteristic or a number field. Therefore, it itself is interesting to examine when $\langle \Delta, \Delta \rangle$ is non-negative in positive characteristic. There are also other interesting problems around the height. For details, see Zhang's paper [13].

From now on, let K be the function field of a smooth projective curve over k, as we only consider the function field case in this article. A key tool to calculate the height $\langle \Delta, \Delta \rangle$ is a formula of Zhang proved in [13]. It tells us that the height can be described in terms of the dualizing sheaf and some invariants arising from the reduction graphs. Let us recall it here. For each $y \in B(k)$, Zhang defined an invariant $\varphi(X_y)$ which depends only on the reduction graph with the associated polarization at y, and proved the formula:

$$\langle \Delta, \Delta \rangle = \frac{2g+1}{2g-2}(\omega_a, \omega_a) - \sum_{y \in B(k)} \varphi(X_y),$$

where ω_a is the admissible dualizing sheaf of X and (ω_a, ω_a) is the admissible pairing in [12]. Here assume that X has a semistable model $f : \mathcal{X} \to B$. Then we know, originally by [12], that

$$(\omega_a, \omega_a) = \left(\omega_{\mathcal{X}/B} \cdot \omega_{\mathcal{X}/B}\right) - \sum_{y \in B(k)} \epsilon(X_y), \tag{0.0.0}$$

where $\omega_{\mathcal{X}/B}$ is the relative dualizing sheaf of f and $\epsilon(X_y)$ is the invariant, which we recall in § 1.10. The formula tells us that it is of importance to compute the self-intersection number of the dualizing sheaf and the graph invariants in order to know the height. Explicit calculation of the graph invariants will be our main work in this note.

In the rest, let us briefly describe the structure of this article together with our results.

In § 1, we will recall the terminology on graphs and introduce some graph invariants. The notion of contraction of edges and the invariants φ and ψ will be of significance. Finally in this section, we recall Zhang's formula, which will play a key role for the calculation of the height.

In § 2, we compute explicitly a certain invariant concerned with φ and ψ for the graphs of genus 3. Using that, we find a sufficient condition for $\langle \Delta, \Delta \rangle$ being positive. Actually Corollary 2.8 says, in any characteristic, that a curve over K without certain kind of reduction graph has the Gross-Schoen cycle with positive height. It will be applied in the last section.

§ 3 will be rather a preparation for the last section. We will calculate the graph invariant ψ for those so-called hyperelliptic polarized graphs (cf. Theorem 3.5).

We will introduce the notion of graphically hyperelliptic curves in § 4. Roughly speaking, X is said to be graphically hyperelliptic if its any polarized metrized reduction graph is same as that of a hyperelliptic curve. Using Theorem 3.5, we can calculate the graph invariant for graphically hyperelliptic curves, which will show Theorem 4.2. Note that this theorem says in particular that for a graphically hyperelliptic curve X, the Cornalba-Harris inequality is equivalent to $\langle \Delta, \Delta \rangle = 0$. Since a hyperelliptic curve is a graphically hyperelliptic curve for which the Cornalba-Harris equality holds true, we will thus give an alternative proof of the fact that $\langle \Delta, \Delta \rangle = 0$ holds for hyperelliptic curves (cf. Corollary 4.3), which is known from the homological triviality of Δ (cf. [5, Corollary 4.9]).

As mentioned just above, if X is a hyperelliptic curve, then it is a graphically hyperelliptic curve with $\langle \Delta, \Delta \rangle = 0$. Now a natural question arises— does the converse hold true? We will propose Conjecture 4.5 insisting it be true, and prove it actually true in the case of genus 3:

Theorem (cf. Theorem 4.6)**.** Let X be a smooth projective curve of genus 3 over a function field. Then X is a hyperelliptic curve if and only if X is a graphically hyperelliptic curve and $\langle \Delta, \Delta \rangle = 0$.

In our proof of it, not only the results of §3 but also that of §2 will be used.

Notation. Let k be a fixed algebraically closed field, K the function field of a geometrically connected non-singular projective curve B over k.

1. Graph invariants and Zhang's formula

First of all, we fix our terminology on graphs. Most of them follow [11] and [4]. Next, we recall some graph invariants due to Zhang. Finally in this section, we recall Zhang's formula.

1.1. **Weighted polarized graphs.** A *graph* G means a triple consisting of a finite set $\mathrm{Vert}(G)$ of vertices, a finite set $\mathrm{Ed}(G)$ of edges, and incidence relations. For each vertex v, let b_v denote the valence at v. A function $q : \mathrm{Vert}(G) \to \mathbb{Z}_{\geq 0}$ such that the divisor

$$K_q := \sum_{v \in \mathrm{Vert}(G)} (2q(v) + b_v - 2)$$

is effective, is called a *polarization* of G. The divisor K_q is called the *canonical divisor* of a polarized graph $\overline{G} = (G, q)$.

The notion of polarization above is essentially the one dealt with in [4]. The objects called polarization in Moriwaki's papers and the author's ones are rather the canonical divisors here.

Let (G, q) be a polarized graph. A vertex v is said to be *eliminable* if $b_v = 2$ and $q(v) = 0$. We can eliminate such eliminable vertices as we like when we consider the graph invariants as we will see later.

Let $b_1(G)$ denote the first Betti number of G. We define the *genus* g of the polarized graph (G, q) by

$$g := b_1(G) + \sum_{v \in \mathrm{Vert}(G)} q(v).$$

It is an invariant for a *polarized* graph.

Let $\mathcal{W}(G)$ be the dual vector space of the $\mathbb{R}$-vector space with formal basis $\mathrm{Ed}(G)$, and put

$$\mathcal{W}_{>0}(G) := \{\lambda \in \mathcal{W}(G) \mid \lambda(e) > 0 \text{ for any } e \in \mathrm{Ed}(G)\ \}.$$

We call each $\lambda \in \mathcal{W}_{>0}(G)$ a *weight* and a pair (G, λ) a *weighted graph*. For an edge e of a graph G equipped with a weight λ, we call $\lambda(e)$ the *length* of e.

1.2. **Contraction.** For an edge e of a graph G, we can construct another graph by contracting e to one point. More generally, for a set of edges S of G, we can construct the *contraction* G_S by contracting all the edges in S (cf. [11, §1.1]). We have a natural surjective map

$$\mathrm{contr}_S : \mathrm{Vert}(G) \to \mathrm{Vert}\,(G_S) \tag{1.0.1}$$

as well as a natural injective map

$$\mathrm{Ed}\,(G_S) \to \mathrm{Ed}\,(G)\,. \tag{1.0.2}$$

Note that the image of (1.0.2) is $\mathrm{Ed}\,(G) \setminus S$. Putting $s := \#S$, we can write $S = \{e_1, \ldots, e_s\}$, and further we put $S_i = \{e_1, \ldots, e_i\}$ for $i = 1, \ldots, s$. Then we can see

$$G_{S_{i+1}} = (G_{S_i})_{\{e_{i+1}\}}$$

if e_{i+1} is regarded as an edge of G_{S_i} via (1.0.2). Thus any contraction of edges can be expressed as successive contractions of one edge.

Suppose that G is equipped with a polarization q. We define the polarization q_S on the contraction G_S as follows. First let us consider the case of $S = \{e_1\}$. If e_1 is a self-loop, then $\mathrm{Vert}(G) = \mathrm{Vert}\,(G_S)$ via (1.0.1), and we put

$$q_S(v) := \begin{cases} q(v) + 1 & \text{if } v \text{ is the extremity of } e_1, \\ q(v) & \text{otherwise.} \end{cases}$$

If it is not a self-loop but a line segment, then the surjective map contr_S in (1.0.1) is bijective except at the extremities w_1, w_2 of e_1. In this case, we define q_S by

$$\begin{cases} q_S(\mathrm{contr}_S(v')) = q_S(w_1) + q_S(w_2) & \text{if } v' \text{ is } w_1 \text{ or } w_2, \\ q_S(\mathrm{contr}_S(v')) = q_S(v') & \text{otherwise}\,. \end{cases}$$

Thus we have defined q_S when $S = \{e_1\}$. The genus of (G_S, q_S) coincides with that of (G, q).

For a general $S = \{e_1, \ldots, e_s\}$, the contraction G_S is obtained by contractions of $e_1, \ldots, e_s$ successively and we can define q_S by induction. It does not depend on the choice of the numbering of the edges. We call (G_S, q_S) the contraction of S of the polarized graph (G, q). It appears by induction that (G_S, q_S) and (G, q) have the same genus.

If G is equipped with a weight λ, we can induce a weight $\lambda|_{G_S}$ on G_S by $\lambda|_{G_S}(e) = \lambda(e)$ via the inclusion (1.0.2). We usually call $(G_S, q_S, \lambda|_{G_S})$ the contraction of S of (G, q, λ).

1.3. **Irreducible components.** Let G_1 and G_2 be subgraphs of G. We say that G is a *one-point sum* of G_1 and G_2 if $G = G_1 \cup G_2$ and if $G_1 \cap G_2$ is a one-point graph. We write $G = G_1 \vee G_2$ for it. A graph is said to be *reducible* if it is a one-point sum of non-trivial subgraphs, and *irreducible* if it is not reducible. Any graph G has an irreducible decomposition: $G = (\cdots((G_1 \vee G_2) \vee G_3) \vee \cdots \vee G_n)$. We usually simply write

$$G = G_1 \vee G_2 \vee \cdots \vee G_n,$$

and call $G_1, \ldots, G_n$ the *irreducible components* of G.

For each irreducible component G_i, put $S_i := \mathrm{Ed}(G_i)$ here. We should note that there is a natural isomorphism $G_i \cong G_{\mathrm{Ed}(G)\setminus S_i}$. Taking account of this identification, we define the *irreducible components of a polarized graph* (G, q) to be the contractions of $\mathrm{Ed}(G) \setminus S_i$ for $i = 1, \ldots, n$. We will find later that it is a reasonable definition from the viewpoint of the graph invariants.

Note that if λ is a weight on G, we can induce the weight $\lambda|_{G_i}$ for each i via the inclusion $\mathrm{Ed}(G_i) \hookrightarrow \mathrm{Ed}(G)$.

1.4. **Realization.** For a weighted graph (G, λ), there exists a metrized graph Γ_λ, called the *realization*, with the following properties:

(a) The graph G naturally gives the data of a finite cell decomposition of Γ_λ such that the vertices correspond to the 0-cells and edges do to the 1-cells.
(b) The length of e as a metrized subspace of Γ_λ is equal to $\lambda(e)$.

If q is a polarization on G, it canonically induces a polarization on the realization in the very sense of [4, §2.1].

For two weighted graphs (G_1, λ_1) and (G_2, λ_2), we say (G_1, λ_1) is *equivalent* to (G_2, λ_2) if their realizations are isometric to each other. For polarizations q_1 on G_1 and q_2 on G_2, we say (G_1, q_1, λ_1) is *equivalent* to (G_2, q_2, λ_2) if there is an isometry between a realization of (G_1, λ_1) and that of (G_2, λ_2) which preserves the polarizations. Furthermore, we say (G_1, q_1) is *equivalent* to (G_2, q_2) if (G_1, q_1, λ_1) is equivalent to (G_2, q_2, λ_2) for some weights λ_1 and λ_2.

Remark 1.1. Any (weighted) polarized graph is equivalent to a unique (weighted) polarized graph without eliminable vertices.

1.5. **Harmonic analysis on a polarized metrized graph.** Let us recall the Green function on a metrized graph due to Zhang. See [12] for details.

Let Γ be a connected metrized graph and let μ be an arbitrary Borel measure on Γ with total volume 1. Then, there exists a unique function $g_\mu(x, y)$ on $\Gamma \times \Gamma$ satisfying the following conditions.

(a) g_μ is continuous, piecewise smooth in both x and y and symmetric in x and y.
(b) For a fixed x, regard $g_\mu(x, y)$ as a function of y, and we have

$$\Delta g_\mu = \delta_x - \mu,$$
$$\int_\Gamma g_\mu \mu = 0.$$

We call this function g_μ the *Green function* for μ.

Let $K = \sum_v a_v v$ be an $\mathbb{R}$-divisor on Γ. If $\deg(K) \neq -2$, then there exists a unique measure $\mu_{(\Gamma,K)}$ of total volume 1 on Γ such that

$$g_{\mu_{(\Gamma,K)}}(K,y) + g_{\mu_{(\Gamma,K)}}(y,y) \tag{1.1.3}$$

is a constant function on y, where $g_{\mu_{(\Gamma,K)}}(K,y) := \sum_v a_v g_{\mu_{(\Gamma,K)}}(v,y)$. We call this measure $\mu_{(\Gamma,K)}$ the *admissible metric* of (Γ, K) and call $g_{\mu_{(\Gamma,K)}}$ the *admissible Green function*. Since it is determined only from (Γ, K), we may write $g_{(\Gamma,K)}$ for $g_{\mu_{(\Gamma,K)}}$. We denote the constant (1.1.3) by $c(\Gamma, K)$ and set

$$\epsilon(\Gamma, K) := 2\deg(K)c(\Gamma, K) - g_{(\Gamma,K)}(K, K).$$

We call this number the *admissible constant* of (Γ, K).

Remark 1.2. Suppose that K is the canonical divisor of a polarized graph of genus g. Then $\deg(K) = 2g - 2$. By its definition, we have

$$c(\Gamma, K) = g_{\mu_{(\Gamma,K)}}(K, y) + g_{\mu_{(\Gamma,K)}}(y, y).$$

Integrating it with respect to δ_K, we have

$$\deg(K)c(\Gamma, K) = g_{\mu_{(\Gamma,K)}}(K, K) + \int g_{\mu_{(\Gamma,K)}}(y, y)\delta_K.$$

By integration with respect to $\deg(K)\mu_{(\Gamma,K)}$ on the other hand, we see

$$\deg(K)c(\Gamma, K) = \int g_{\mu_{(\Gamma,K)}}(y, y)\deg(K)\mu_{(\Gamma,K)}.$$

Accordingly we find

$$\epsilon(\Gamma, K) = \int g_{\mu_{(\Gamma,K)}}(y, y)\delta_K + \int g_{\mu_{(\Gamma,K)}}(y, y)\deg(K)\mu_{(\Gamma,K)} = \int g_{\mu_{(\Gamma,K)}}(y, y)((2g-2)\mu + \delta_K),$$

which is nothing but the invariant $\epsilon(\Gamma)$ in [13, §4.1].

1.6. **Graph invariants (I).** We introduce some invariants arising from graphs. For our latter purpose, we describe them as *functions on the weights*.

Let $\overline{G} = (G, q)$ be a polarized graph of genus g. For each $e \in \mathrm{Ed}(G)$, we can assign an integer i called the *type* in the following way. Let $(G^{\{e\}}, q^{\{e\}})$ be the contraction of $\mathrm{Ed}(G)\setminus\{e\}$, which is a polarized graph consisting of one edge. If it is a self-loop, then we put $i := 0$. If it is a line segment, let v and w be the extremities, and we put $i := \min\{q^{\{e\}}(v), q^{\{e\}}(w)\}$. Since $q^{\{e\}}(v) + q^{\{e\}}(w) = g$, we have $0 \leq i \leq [g/2]$. We denote by $\mathrm{Ed}_i(\overline{G})$ the set of edges of G of type i. Then we define a function $\delta_i(\overline{G})$ on $\mathcal{W}_{>0}(G)$ by

$$\delta_i(\overline{G})(\lambda) := \sum_{e \in \mathrm{Ed}_i(\overline{G})} \lambda(e)$$

for each $\lambda \in \mathcal{W}_{>0}(G)$. Further we put $\delta(\overline{G}) = \sum_i \delta_i(\overline{G})$. Note that $\delta(\overline{G})(\lambda)$ is nothing but the total length of the realization of $(\overline{G}, \lambda)$.

Next, let us define $r_G(v, w)$ so called the *effective resistance*. Let v, w be vertices of G. Let λ be a weight on G and let Γ_λ be the realization. Let δ_v be the Dirac measure supported at v and consider the Green function g_{δ_v}. Now we define

$$r_G(v, w) : \mathcal{W}_{>0}(G) \to \mathbb{R}$$

by $r_G(v,w)(\lambda) = g_{\delta_v}(w,w)$.

Some comments on $r_G(v,w)$ should be added. A weighted graph can be regarded as an electric circuit in a usual way. Then $r_G(v,w)(\lambda)$ is nothing but the electric resistance between v and w (cf. [12, Proposition 3.3]).[1] Therefore, if G is just a line segment for example, and v and w are the extremities of G, then $r_G(v,w)(\lambda)$ is the length of G.

Finally in this subsection, we define $r(\overline{G})$ as follows: Let $K_q = \sum_v d_v v$ be the canonical divisor of $\overline{G}$. Then we put

$$r(\overline{G}) = r_G(K_q, K_q) := \sum_{v,w \in \mathrm{Vert}(G)} d_v d_w r_G(v,w).$$

1.7. **Graph invariants (II).** Let λ be a weight on a polarized graph $\overline{G} = (G,q)$ of genus $g \geq 2$, and let Γ_λ be the realization. Then the canonical divisor $K = K_q$ satisfies the condition $\deg K \neq -2$ and hence we can consider the admissible Green function and the values $g_{(\Gamma_\lambda, K)}(v,w)$ for all $v, w \in \mathrm{Vert}(G)$. We define a function $g_{\overline{G}}(v,w)$ on $\mathcal{W}_{>0}(G)$ by

$$g_{\overline{G}}(v,w)(\lambda) := g_{(\Gamma_\lambda, K)}(v,w).$$

From the definition of the admissible Green function, the function

$$(g_{\overline{G}}(K,v) + g_{\overline{G}}(v,v)) : \mathcal{W}_{>0}(G) \to \mathbb{R}$$

is independent of the choice of v (cf. (1.1.3)), and hence we can define a function $c(\overline{G})$ on $\mathcal{W}_{>0}(G)$ to be it. Further we put

$$\epsilon(\overline{G}) := 2\deg(K)c(\overline{G}) - g_{\overline{G}}(K,K),$$

which is also a function on $\mathcal{W}_{>0}(G)$ such that $\epsilon(\overline{G})(\lambda)$ is the admissible constant of the realization (Γ_λ, K).

Now we define a function $\varphi(\overline{G})$ by

$$\varphi(\overline{G}) := \frac{5g-2}{4(g-1)}\epsilon(\overline{G}) - \frac{3}{8(g-1)}r(\overline{G}) - \frac{1}{4}\delta(\overline{G}).$$

By virtue of Remark 1.2 and [4, Corollary 2.4], $\varphi(\overline{G})(\lambda)$ coincides with the φ for the realization of $(\overline{G}, \lambda)$ dealt with in [13]. Further we put

$$\begin{aligned}\psi(\overline{G}) : &= \epsilon(\overline{G}) + \frac{2g-2}{2g+1}\varphi(\overline{G}) \\ &= \frac{9g}{2(2g+1)}\epsilon(\overline{G}) - \frac{3}{4(2g+1)}r(\overline{G}) - \frac{g-1}{2(2g+1)}\delta(\overline{G}).\end{aligned}$$

As we will see later, these invariants is closely related to the height of the canonical Gross-Schoen cycle.

[1] The exposition [1, §6] would be a good reference. With their notation, the "voltage function $j_z(x,y)$" stands for our function $g_{\delta_z}(x,y)$.

1.8. **The contraction lemma and the sum formula.** Let us recall formulae which play important roles in calculating the graph invariants. Suppose that for any polarized graph $\overline{G}$, we are given a function $F(\overline{G})$ on $\mathcal{W}_{>0}(G)$. We say that the *contraction lemma* holds for F if for any $\overline{G}$ and $e \in \mathrm{Ed}(G)$, we have

$$F\left(\overline{G}_{\{e\}}\right)\left(\lambda|_{G_{\{e\}}}\right) = \lim_{\lambda(e)\to 0} F(\overline{G})(\lambda).$$

Next let $\overline{G}_1, \dots, \overline{G}_n$ be the irreducible components of $\overline{G}$. We say that the *sum formula* holds for F if it has the property that

$$F(\overline{G})(\lambda) = F(\overline{G}_1)(\lambda|_{G_1}) + \cdots + F(\overline{G}_n)(\lambda|_{G_n})$$

for any weight λ on G.

By virtue of [8, Lemma 3.1] and [11, Proposition 1.10], we find that the contraction lemma and the sum formula hold for φ and ψ.

1.9. **The invariant ψ for polarized trees.** Let us here calculate $\psi(\overline{G})$ when $\overline{G} = (G, q)$ is a polarized tree of genus $g \geq 2$, for example.

First assume G to consist of a unique edge e, with the extremities v and w. We put $q(v) = i$ and hence $q(w) = g - i$. Then we have $K_q = (2i-1)v + (2(g-i)-1)w$. Let λ be a weight on G. By virtue of [6, Proposition 4.5], we know

$$\epsilon(\overline{G})(\lambda) = \left(\frac{4i(g-i)}{g} - 1\right)\lambda(e),$$

where recall that $\lambda(e)$ is the length of e. By the definition of $r(\overline{G})$, we see

$$r(\overline{G})(\lambda) = 2(2i-1)(2(g-i)-1)\lambda(e).$$

Taking account that $\lambda(e) = \delta_i(\overline{G})(\lambda) = \delta(\overline{G})(\lambda)$ holds in this case, we can directly obtain

$$\psi(\overline{G})(\lambda) = \left(\frac{12i(g-i)}{2g+1} - 1\right)\delta_i(\overline{G})(\lambda). \tag{1.2.4}$$

Next let us consider the general case, namely, let $\overline{G} = (G, q)$ be a polarized graph such that G is a tree. Note that any edge of G itself is an irreducible component of G, and let $\overline{G}^{\{e\}}$ be the contraction of all edges of G but e. Then, by the sum formula, we have

$$\psi(\overline{G})(\lambda) = \sum_{e \in \mathrm{Ed}(G)} \psi\left(\overline{G}^{\{e\}}\right)\left(\lambda|_{G^{\{e\}}}\right),$$

where $\lambda|_{G^{\{e\}}}$ is the weight on $G^{\{e\}}$ induced from λ. Accordingly, by (1.2.4), we have

$$\psi(\overline{G}) = \sum_{i=1}^{[\frac{g}{2}]} \left(\frac{12i(g-i)}{2g+1} - 1\right)\delta_i(\overline{G}). \tag{1.2.5}$$

1.10. **The dual graphs and the invariants.** Suppose that $f : \mathcal{X} \to B$ is a semistable model of a smooth projective curve X of genus $g \geq 2$. For a closed point $y \in B$, let $\mathcal{X}_y$ denote the fiber of f over y. It is well-known that the graph G_y by configuration of $\mathcal{X}_y$ can be defined so that the vertices correspond to the irreducible components of $\mathcal{X}_y$ and the edges do to the nodes. We call it the *dual graph* over y. The dual graph has a canonical polarization q_y defined by

$$q_y(v) := (\text{the geometric genus of the irreducible component corresponding to } v)$$

for $v \in \mathrm{Vert}(G_y)$. We call $\overline{G}_y = (G_y, q_y)$ the *polarized dual graph* over y. The canonical divisor of $\overline{G}_y$ has degree $2g-2$.

We can also construct a natural weight λ_y on G_y in the following way. Let P be a node of $\mathcal{X}_y$. Then the completion $\hat{\mathcal{O}}_{\mathcal{X},P}$ of the local ring at P in $\mathcal{X}$ is of form $k[[u,v]]/(uv - t^m)$, where t is a local parameter at y on B and m is a natural number. We call this m the thickness of the node P, and define λ_y by

$$\lambda_y(e) := (\text{the thickness of the node corresponding to } e).$$

Thus we have the *weighted polarized graph* $(\overline{G}_y, \lambda_y)$ over y.

Although $(\overline{G}_y, \lambda_y)$ is defined after the choice of semistable models, it should be noted that it is uniquely determined up to equivalence. In particular, the graph invariants for $(\overline{G}_y, \lambda_y)$ on which we are focusing depend only on X. Accordingly we may simply write

$$\epsilon(X_y) := \epsilon(\overline{G}_y)(\lambda_y), \quad \varphi(X_y) := \varphi(\overline{G}_y)(\lambda_y), \quad \psi(X_y) := \psi(\overline{G}_y)(\lambda_y), \quad \delta_i(X_y) := \delta_i(\overline{G}_y)(\lambda_y).$$

Note that if $\mathcal{X}$ is nonsingular, then $\delta_i(X_y)$ is nothing but the number of nodes of type i in the fiber $\mathcal{X}_y$.

1.11. **Zhang's formula.** Let $\langle \cdot, \cdot \rangle$ be the height pairing studied in [5] and [13]. We call $\langle \Delta, \Delta \rangle$ the height of the canonical Gross-Schoen cycle Δ. Here we repeat Zhang's formula in [13, Corollary 1.3.2]. In our situation, it says

$$\langle \Delta, \Delta \rangle = \frac{2g+1}{2g-2}(\omega_a, \omega_a) - \sum_y \varphi(\overline{G}_y)(\lambda_y).$$

Assume that X has a semistable model $f : \mathcal{X} \to B$. Taking account of (0.0.0) and the definition of ψ, we have

$$\langle \Delta, \Delta \rangle = \frac{2g+1}{2g-2}\left((\omega_{\mathcal{X}/B} \cdot \omega_{\mathcal{X}/B}) - \sum_y \psi(\overline{G}_y)(\lambda_y) \right), \tag{1.2.6}$$

which will be the fundamental formula in our study of the height.

2. The height of the Gross-Schoen cycle of curves of genus 3

In this section, we consider the positivity of the height of the Gross-Schoen cycle of non-hyperelliptic curves of genus 3 by using Zhang's formula.

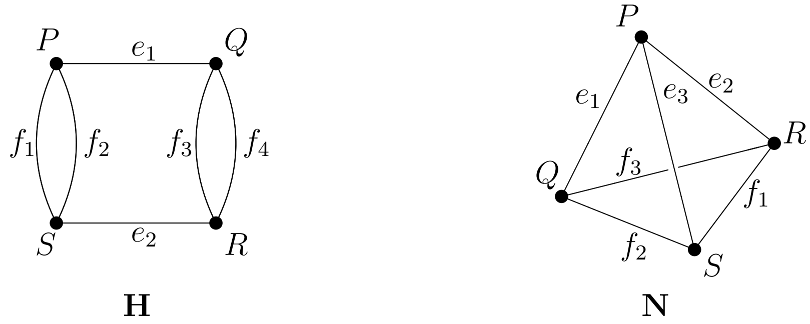

FIGURE 1. The maximal models.

2.1. **Remarks on polarized graph of genus** 3**.** We recall here some notions and facts on polarized graphs of genus 3. See [8] for details.

First let us consider the graphs $\mathbf{H}$ and $\mathbf{N}$ as in Figure 1. The polarized graphs $\overline{\mathbf{H}} = (\mathbf{H}, 0)$ and $\overline{\mathbf{N}} = (\mathbf{N}, 0)$ are called *maximal models*, where 0 stands for the polarization which is the constant function 0. They are irreducible polarized graphs of genus 3 without eliminable vertices.

Let $\overline{M}$ be a maximal model, i.e., $\overline{\mathbf{H}}$ or $\overline{\mathbf{N}}$. We say $\overline{M}$ *is a maximal model of* $\overline{G}$ if $\overline{G}$ is equivalent to the contraction of edges of $\overline{M}$. It is not difficult to see that any polarized graph $\overline{G}$ of genus 3 with only edges of type 0 is equivalent to those polarized graphs having $\overline{\mathbf{H}}$ or $\overline{\mathbf{N}}$ as a maximal model. Moreover, if $\overline{G}$ is not equivalent to $\overline{\mathbf{N}}$, it must have $\overline{\mathbf{H}}$ as a maximal model.

Let $\overline{G} = (G, q)$ be a polarized graph of genus 3 *without eliminable vertices*. We recall, for a pair of edges of G, the notion of *h-type* introduced in [8]. Let e and e' be distinct edges of G. Let $\overline{G}^{\{e,e'\}}$ be the contraction of all edges but e, e'. We say that the pair $\{e, e'\}$ of edges is of *h-type* if it satisfies the following conditions:

(a) $\overline{G}^{\{e,e'\}}$ is an irreducible graph with two vertices, say v, w.
(b) The induced polarization $q^{\{e,e'\}}$ is of form $q^{\{e,e'\}}(v) = q^{\{e,e'\}}(w) = 1$.

For example, the set $\{e_1, e_2\}$ of edges of $\mathbf{H}$ in Figure 1 is a pair of h-type of $\overline{\mathbf{H}}$.

Lemma 2.1. *A polarized graph* (G, q) *of genus* 3 *without eliminable vertices has at most one pair of edges of h-type.*

Proof. Let $\{e_1, e_2\}$ be a pair of edges of h-type. Note that e_1 and e_2 sit in the same irreducible component, since otherwise the contraction of the edges other than $\{e_1, e_2\}$ must be a reducible graph.

Let $\{e_1', e_2'\}$ be also a pair of h-type. Suppose that $\{e_1', e_2'\}$ and $\{e_1, e_2\}$ sit in different irreducible components. Contracting all the edges other than $\{e_1, e_2, e_1', e_2'\}$, we obtain a polarized graph as Figure 2. Since it has the first Betti number 2, the polarization must have 0 at some vertex with valence 2, which contradicts our assumption that it does not have an eliminable vertex.

Accordingly, we see that $\{e_1', e_2'\}$ and $\{e_1, e_2\}$ sit in the same irreducible component, and hence we may assume G is irreducible. First, it is obvious by the Figure 1 that a maximal model can have at most one pair of h-type. In general, let $\overline{M}$ be a maximal model of $\overline{G}$.

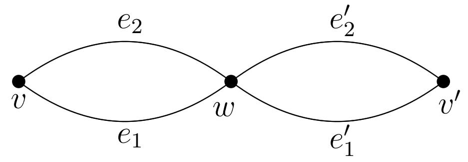

FIGURE 2

Since G is a contraction of edges of M, we can regard $\mathrm{Ed}(G) \subset \mathrm{Ed}(M)$ and hence $\{e_1', e_2'\}$ and $\{e_1, e_2\}$ are pair of edges of h-type of $\overline{M}$. Since $\overline{M}$ is maximal, we have $\{e_1', e_2'\} = \{e_1, e_2\}$ as we have already known. □

Let $\overline{G}$ be a polarized graph of genus 3 (which may have an eliminable vertex). We define $h(\overline{G})$, as a function on $\mathcal{W}_{>0}(G)$, as follows: First we consider the case where G does not have an eliminable vertex. If $\overline{G}$ has a pair $\{e, e'\}$ of h-type, then put

$$h(\overline{G})(\lambda) := \min\{\lambda(e), \lambda(e')\},$$

and otherwise we put $h(\overline{G}) = 0$. Next consider it for general $\overline{G}$. For a weight $\lambda \in \mathcal{W}_{>0}(G)$, we can take uniquely a weighted polarized graph $(\overline{G}', \lambda')$ equivalent to $(\overline{G}, \lambda)$ such that $\overline{G}'$ does not have an eliminable vertex, as noted in Remark 1.1. Then we define $h(\overline{G})(\lambda) := h(\overline{G}')(\lambda')$.

2.2. **A graph invariant Φ and the height.** Now we define, for a polarized graph $\overline{G}$ of genus 3 with edges of type 0 only, an invariant Φ by

$$\Phi(\overline{G}) := \frac{1}{3}\delta_0(\overline{G}) + \frac{4}{3}h(\overline{G}) - \psi(\overline{G}).$$

From the definition, we see that the contraction lemma and the sum formula hold for Φ.

The following assertion is the first step for our purpose.

Proposition 2.2. *Let $f : \mathcal{X} \to B$ be a non-hyperelliptic semistable curve of genus 3 with the smooth generic fiber X. For each critical value $y \in B$ of f, let $(\overline{G}_y, \lambda_y)$ be the weighted polarized dual graph over y and let $(\overline{G}_{y0}, \lambda_{y0})$ be the contraction of all edges of $\overline{G}_y$ of type 1. Then we have*

$$\langle \Delta, \Delta \rangle \geq \frac{7}{4} \sum_y \Phi(\overline{G}_{y0})(\lambda_{y0}) + \delta_1(X),$$

where $\delta_1(X) := \sum_y \delta_1(X_y)$.

Proof. We know, as (1.2.6),

$$\langle \Delta, \Delta \rangle = \frac{7}{4}\left((\omega_{\mathcal{X}/B} \cdot \omega_{\mathcal{X}/B}) - \sum_{y \in B(k)} \psi_y(\overline{G}_y)(\lambda_y) \right).$$

Since f is non-hyperelliptic, we have by [8, Corollary 3.8]

$$(\omega_{\mathcal{X}/B} \cdot \omega_{\mathcal{X}/B}) \geq \sum_y \left(\frac{1}{3}\delta_0(\overline{G}_y)(\lambda_y) + 3\delta_1(\overline{G}_y)(\lambda_y) + \frac{4}{3}h(\overline{G}_y)(\lambda_y) \right).$$

Let $\overline{G}_{y1}$ be the contraction of all the edges of type 0 of $\overline{G}_y$ and let λ_{y1} be the weight on $\overline{G}_{y1}$ induced from λ_y. Since any irreducible component of $\overline{G}_{y1}$ has only one type of edges, we have

$$\psi_y(\overline{G}_y)(\lambda_y) = \psi(\overline{G}_{y0})(\lambda_{y0}) + \psi(\overline{G}_{y1})(\lambda_{y1})$$

by the sum formula. Accordingly we find

$$\begin{aligned}
&\left(\omega_{\mathcal{X}/B} \cdot \omega_{\mathcal{X}/B}\right) - \sum_y \psi_y(\overline{G}_y)(\lambda_y) \\
&\geq \sum_y \left(\left(\frac{1}{3}\delta_0(\overline{G}_{y0})(\lambda_{y0}) + \frac{4}{3}h(\overline{G}_{y0})(\lambda_{y0}) - \psi(\overline{G}_{y0})(\lambda_{y0}) \right) + \left(3\delta_1(\overline{G}_{y1})(\lambda_{y1}) - \psi(\overline{G}_{y1})(\lambda_{y1})\right) \right) \\
&= \sum_y \Phi(\overline{G}_{y0})(\lambda_{y0}) + \sum_y \left(3\delta_1(\overline{G}_{y1})(\lambda_{y1}) - \psi(\overline{G}_{y1})(\lambda_{y1})\right).
\end{aligned}$$

Since $\overline{G}_{y1}$ is a polarized tree, we have

$$\psi(\overline{G}_{y1}) = \frac{17}{7}\delta_1(\overline{G}_{y1})$$

by (1.2.5), and hence

$$3\delta_1(\overline{G}_{y1})(\lambda_{y1}) - \psi(\overline{G}_{y1})(\lambda_{y1}) = \frac{4}{7}\delta_1(\overline{G}_{y1})(\lambda_{y1}) = \frac{4}{7}\delta_1(\overline{G}_y)(\lambda_y).$$

Thus we have our inequality. □

2.3. **Estimate of Φ and the results.** We know that the positivity of the height follows from that of the invariant Φ for the dual graphs by virtue of Proposition 2.2. In this subsection, we describe $\Phi(\overline{G})$ explicitly, examine whether it is positive or not and obtain some results.

Let us give explicit description for $\Phi(\overline{G})$ first:

Proposition 2.3. *Let $\overline{G}$ be a maximal model of genus 3 (cf. Figure 1).*

(1) *Suppose $\overline{G} = \overline{\mathbf{H}}$. For a weight λ on $\mathbf{H}$, put $l_i = \lambda(e_i)$ for $i = 1, 2$ and $m_i = \lambda(f_i)$ for $i = 1, \ldots, 4$, and let σ_k, for each natural number k, be the k-th elementary symmetric polynomial on $m_1, \ldots, m_4$. We put further*

$$L := (l_1 + l_2)(m_1 + m_2)(m_3 + m_4) + \sigma_3.$$

Then we have

$$\begin{aligned}
\Phi(\overline{\mathbf{H}})(\lambda) = \frac{1}{21}(l_1 + l_2 + m_1 + m_2 + m_3 + m_4) + \frac{4}{3}\min\{l_1, l_2\} \\
- \frac{12}{7}\frac{l_1 l_2 (m_1 + m_2)(m_3 + m_4)}{L} - \frac{3}{7}\frac{(l_1 + l_2)\sigma_3}{L}.
\end{aligned}$$

(2) *Suppose $\overline{G} = \overline{\mathbf{N}}$. For a weight λ on $\mathbf{N}$, put $l_i = \lambda(e_i)$ and $m_i = \lambda(f_i)$ for $i = 1, 2, 3$, and let σ'_k, for each natural number k, be the k-th elementary symmetric polynomial on $l_1, l_2, l_3, m_1, m_2, m_3$. We put further*

$$L' := \sigma'_3 - (l_1 l_2 l_3 + l_1 m_2 m_3 + l_3 m_1 m_2).$$

Then we have

$$\Phi(\overline{\mathbf{N}})(\lambda) = \frac{1}{21}\sigma_1' - \frac{3}{7}\frac{\sigma_4' - (l_1 l_2 m_1 m_2 + l_2 l_3 m_2 m_3 + l_3 l_1 m_3 m_1)}{L'}.$$

Proof. As in [8, Proposition 3.1], we have already known the explicit formula for the admissible constant $\epsilon(\overline{G})$. By messy but elementary calculation, we find

$$r(\overline{G})(\lambda) = \begin{cases} \dfrac{8l_1 l_2(m_1+m_2)(m_3+m_4) + 6(l_1+l_2)\sigma_3 + 8\sigma_4}{L} & \text{if } \overline{G} = \overline{\mathbf{H}}, \\[2ex] \dfrac{6\sigma_4' + 2(l_1 l_2 m_1 m_2 + l_2 l_3 m_2 m_3 + l_3 l_1 m_3 m_1)}{L'} & \text{if } \overline{G} = \overline{\mathbf{N}}. \end{cases}$$

From the definition of $h(\overline{G})$, we have

$$h(\overline{G})(\lambda) = \begin{cases} \min\{l_1, l_2\} & \text{if } \overline{G} = \overline{\mathbf{H}}, \\ 0 & \text{if } \overline{G} = \overline{\mathbf{N}}. \end{cases}$$

Accordingly, we can obtain our formulae immediately from the definition of $\Phi(\overline{G})$. □

By virtue of the contraction lemma, we have also the following formulae.

Corollary 2.4. *Let $\overline{G} = (G, q)$ be a polarized graph of genus 3 without eliminable vertices, and let λ be a weight on G.*

(1) *Suppose that G is irreducible and has 3 edges and 2 vertices (cf. $\mathbf{E}_2$ in Figure 3). Let m_1, m_2, m_3 be the length of the edges. Then we have*

$$\Phi(\overline{G})(\lambda) = \frac{1}{21}(m_1+m_2+m_3) - \frac{3}{7}\frac{m_1 m_2 m_3}{m_1 m_2 + m_2 m_3 + m_3 m_1}.$$

(2) *If G is the sum of 3 loops of length m_1, m_2, m_3, then*

$$\Phi(\overline{G})(\lambda) = \frac{1}{21}(m_1+m_2+m_3).$$

Now let us consider the positivity of $\Phi(\overline{G})$. Suppose that $\overline{G}$ has $\overline{\mathbf{H}}$ as a maximal model. The non-negativity of $\Phi(\overline{G})(\lambda)$ for any $\lambda \in \mathcal{W}_{>0}(G)$ follows from that of $\Phi(\overline{\mathbf{H}})(\lambda)$ for any $\lambda \in \mathcal{W}_{>0}(\mathbf{H})$ by the contraction lemma, but the positivity does not. To examine the positivity, we will use the following lemmas which tell us the condition for $\Phi(\overline{G})(\lambda)$ to be 0. In the proof of Theorem 2.7, Lemma 2.6 will be used for a general case and Lemma 2.5 will be applied for the specialized cases where the denominator of $\Phi(\overline{\mathbf{H}})$ vanishes.

Lemma 2.5. *Let l, m_1, m_2, m_3 be non-negative real numbers. Suppose that $lm_1 + m_1 m_2 + m_2 l \neq 0$. Then*

$$\frac{1}{21}(l+m_1+m_2+m_3) - \frac{3}{7}\frac{lm_1m_2}{lm_1+m_1m_2+m_2l} \geq 0,$$

and the equality holds if and only if "$m_3 = 0$, and $l = m_1 = m_2$".

Proof. It is enough to show that

$$(l+m_1+m_2)(lm_1+m_1m_2+m_2l) \geq 9lm_1m_2$$

and the equality holds if and only if $l = m_1 = m_2$. That easily follows from the geometric-arithmetic inequality. □

Lemma 2.6. *Let $l_1, l_2, m_1, \ldots, m_4$ be non-negative real numbers. Suppose that one of the following holds:*

(a) $(l_1 + l_2)(m_1 + m_2)(m_3 + m_4) \neq 0$.
(b) $l_1 = l_2 = 0$ *and at most one of* $m_1, \ldots, m_4$ *is* 0.

Then $L \neq 0$ and we have

$$\frac{1}{21}(l_1+l_2+m_1+m_2+m_3+m_4)+\frac{4}{3}\min\{l_1,l_2\}-\frac{12}{7}\frac{l_1l_2(m_1+m_2)(m_3+m_4)}{L}-\frac{3}{7}\frac{(l_1+l_2)\sigma_3}{L}\geq 0,$$

where the notation of Proposition 2.3 (1) is adopted. Moreover, the equality holds if and only if "$m_1 = m_2$, $m_3 = m_4$, $\min\{l_1, l_2\} = 0$ and $\max\{l_1, l_2\} = m_1 + m_3$".

Proof. Step 1. The case $l_2 = 0$.

We put $l := l_1$. Let us consider a quadratic function

$$\begin{aligned} f(l) &:= (l + m_1 + m_2 + m_3 + m_4)(l(m_1 + m_2)(m_3 + m_4) + \sigma_3) - 9l\sigma_3 \\ &= (m_1 + m_2)(m_3 + m_4)l^2 - (8\sigma_3 - \sigma_1(m_1 + m_2)(m_3 + m_4))l + \sigma_1\sigma_3 \end{aligned}$$

on l. Then what we like to examine is the positivity of $f(l)/L$. Accordingly, it is enough to prove that $f(l)$ is non-negative, and is equal to 0 if and only if "$m_1 = m_2$, $m_3 = m_4$, and $l = m_1 + m_3$".

Suppose

$$8\sigma_3 - \sigma_1(m_1 + m_2)(m_3 + m_4) \leq 0.$$

Then, since $(m_1 + m_2)(m_3 + m_4) > 0$ and $\sigma_1\sigma_3 > 0$ by our assumption, we find that $f(l)$ is positive for any $l \geq 0$. Therefore we may assume

$$8\sigma_3 - \sigma_1(m_1 + m_2)(m_3 + m_4) > 0. \tag{2.6.7}$$

Let D be the discriminant of the quadratic $f(l)$. If $D < 0$, then $f(l)$ is positive definite. By a direct calculation, we have

$$\begin{aligned} -D &= 4(m_1 + m_2)(m_3 + m_4)\sigma_1\sigma_3 - (8\sigma_3 - \sigma_1(m_1 + m_2)(m_3 + m_4))^2 \\ &= (m_1 + m_2)(m_3 + m_4)\sigma_1(4\sigma_3 - (\sigma_1(m_1 + m_2)(m_3 + m_4)) \\ &\quad + 16\sigma_3\left((m_1 + m_2)(m_3 + m_4)\sigma_1 - 4\sigma_3\right) \\ &= (16\sigma_3 - (m_1 + m_2)(m_3 + m_4)\sigma_1)\left((m_1 + m_2)(m_3 + m_4)\sigma_1 - 4\sigma_3\right). \end{aligned}$$

It is elementary to check

$$(m_1 + m_2)(m_3 + m_4)\sigma_1 - 4\sigma_3 \geq 0 \tag{2.6.8}$$

and the equality holds if and only if "$m_1 = m_2$ and $m_3 = m_4$". By the assumption (2.6.7), we see that $-D$ is non-negative and is equal to 0 if and only if "$m_1 = m_2$ and $m_3 = m_4$". In this case, we have

$$f(l) = 4m_1m_3(l - (m_1 + m_3))^2,$$

and it is 0 if and only if $l = m_1 + m_3$. Thus we have our assertion in this case.

Step 2. The general case.

Without loss of generality, we may assume $0 \leq l_2 \leq l_1$. Let us write l for l_1 again. We can put $l_2 = \rho l$ for $0 \leq \rho \leq 1$. Put

$$L_\rho := (1 + \rho)l(m_1 + m_2)(m_3 + m_4) + \sigma_3.$$

Our goal is to prove the positivity of

$$((1+\rho)l+\sigma_1)L_\rho + 28\rho l L_\rho - 36\rho l^2(m_1+m_2)(m_3+m_4) - 9(1+\rho)l\sigma_3 \tag{2.6.9}$$

for $0<\rho\le 1$. Now we can see that (2.6.9) is equal to

$$\begin{aligned} g(\rho) := 29l^2(m_1+m_2)(m_3+m_4)\rho^2 \\ - l(6l(m_1+m_2)(m_3+m_4) - 20\sigma_3 - (m_1+m_2)(m_3+m_4)\sigma_1)\rho \\ + l^2(m_1+m_2)(m_3+m_4) - 8l\sigma_3 + l(m_1+m_2)(m_3+m_4)\sigma_1 + \sigma_1\sigma_3. \end{aligned}$$

Note that

$$l^2(m_1+m_2)(m_3+m_4) > 0$$

by our assumption.

Suppose

$$6l(m_1+m_2)(m_3+m_4) - 20\sigma_3 - (m_1+m_2)(m_3+m_4)\sigma_1 \tag{2.6.10}$$

is negative. Then, as a function on $\rho(\ge 0)$, $g(\rho)$ takes its minimum when $\rho=0$. Since $g(0)\ge 0$ by Step 1, we have $g(\rho)>0$ for $\rho>0$. Accordingly we may assume that (2.6.10) is non-negative, namely,

$$l \ge \frac{20\sigma_3 + (m_1+m_2)(m_3+m_4)\sigma_1}{6(m_1+m_2)(m_3+m_4)}. \tag{2.6.11}$$

Let D be the discriminant of the quadratic $g(\rho)$ on ρ and put $h(l) := (-D)/l^2$. If $h(l)>0$, then $D<0$ and hence the quadratic function $g(\rho)$ is positive definite. Thus we are reduced to show $h(l)>0$ under the assumption (2.6.11).

We have

$$\begin{aligned} h(l) = 80(m_1+m_2)^2(m_3+m_4)^2 l^2 \\ - 16(m_1+m_2)(m_3+m_4)(43\sigma_3 - 8(m_1+m_2)(m_3+m_4)\sigma_1)l \\ + 76(m_1+m_2)(m_3+m_4)\sigma_1\sigma_3 - 400{\sigma_3}^2 - (m_1+m_2)^2(m_3+m_4)^2{\sigma_1}^2, \end{aligned}$$

which is a quadratic function on l. The axis of h as a quadratic function on l is given by

$$l = \frac{43\sigma_3 - 8(m_1+m_2)(m_3+m_4)\sigma_1}{10(m_1+m_2)(m_3+m_4)}.$$

Since

$$\begin{aligned} \frac{20\sigma_3 + (m_1+m_2)(m_3+m_4)\sigma_1}{6(m_1+m_2)(m_3+m_4)} - \frac{43\sigma_3 - 8(m_1+m_2)(m_3+m_4)\sigma_1}{10(m_1+m_2)(m_3+m_4)} \\ = \frac{29}{30(m_1+m_2)(m_3+m_4)}((m_1+m_2)(m_3+m_4)\sigma_1 - \sigma_3) \ge 0, \end{aligned}$$

we see that $h(l)$ takes its minimum when

$$l = l_0 := \frac{20\sigma_3 + (m_1+m_2)(m_3+m_4)\sigma_1}{6(m_1+m_2)(m_3+m_4)}.$$

Then

$$\begin{aligned}
h(l_0) = & \frac{20}{9}(20\sigma_3 + (m_1+m_2)(m_3+m_4)\sigma_1)^2 \\
& - \frac{8}{3}(43\sigma_3 - 8(m_1+m_2)(m_3+m_4)\sigma_1)(20\sigma_3 + (m_1+m_2)(m_3+m_4)\sigma_1) \\
& + 76(m_1+m_2)(m_3+m_4)\sigma_1\sigma_3 - 400{\sigma_3}^2 - (m_1+m_2)^2(m_3+m_4)^2{\sigma_1}^2 \\
= & -\frac{16240}{9}{\sigma_3}^2 + \frac{4292}{9}(m_1+m_2)(m_3+m_4)\sigma_1\sigma_3 + \frac{203}{9}(m_1+m_2)^2(m_3+m_4)^2{\sigma_1}^2 \\
= & \frac{4292\sigma_3}{9}((m_1+m_2)(m_3+m_4)\sigma_1 - 4\sigma_3) + \frac{928}{9}{\sigma_3}^2 + \frac{203}{9}(m_1+m_2)^2(m_3+m_4)^2{\sigma_1}^2 \\
> & 0,
\end{aligned}$$

where we use an elementary inequality (2.6.8). Thus we obtain our inequality. □

By virtue of the above lemmas, we can find when $\Phi(\overline{G})(\lambda)$ is positive:

Theorem 2.7. *Let $\overline{G}$ be a polarized graph of genus 3 without eliminable vertices. Suppose that $\overline{\mathbf{H}}$ is a maximal model of $\overline{G}$. Then we have $\Phi(\overline{G})(\lambda) \geq 0$ for any $\lambda \in \mathcal{W}_{>0}(G)$. Moreover, $\Phi(\overline{G})(\lambda) = 0$ if and only if one of the following cases occurs:*

(a) *G is isomorphic to $\mathbf{E}_1$ in Figure 3 and*

$$\lambda(f_1) = \lambda(f_2), \quad \lambda(f_3) = \lambda(f_4), \quad \lambda(e) = \lambda(f_1) + \lambda(f_3).$$

(b) *G is isomorphic to $\mathbf{E}_2$ in Figure 3 and*

$$\lambda(e_1) = \lambda(e_2) = \lambda(e_3).$$

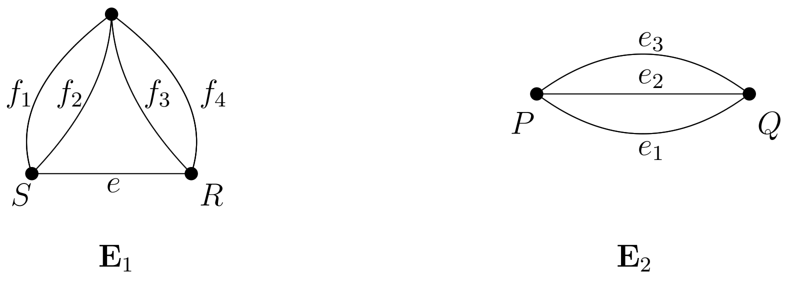

FIGURE 3

Proof. By our assumption, we have $G = \mathbf{H}_S$ for some $S \subset \mathrm{Ed}(\mathbf{H})$. Then the non-negativity is immediate from Proposition 2.3 (1) and Lemma 2.6 since the contraction lemma holds for Φ. Further, Proposition 2.3 (1), Corollary 2.4, Lemma 2.5, Lemma 2.6 and the contraction lemma tell us that $\Phi(\overline{G})(\lambda) = 0$ if and only if (a) or (b) occurs. □

Taking account on Remark 1.1, we obtain the following assertion as an immediate corollary of Proposition 2.2 and Theorem 2.7.

Corollary 2.8 (char$(k) \geq 0$)**.** *Let $f : \mathcal{X} \to B$ be a non-hyperelliptic semistable curve of genus 3 with the smooth generic fiber X, and let $(\overline{G}_y, \lambda_y)$ be the weighted polarized dual graph over $y \in B$. Assume that $\Phi(\overline{G}_y)(\lambda_y) \geq 0$ for all y such that $\overline{G}_y$ is equivalent to $\overline{\mathbf{N}}$. Then we have $\langle \Delta, \Delta \rangle \geq 0$. In addition, suppose that not all $(\overline{G}_y, \lambda_y)$ are equivalent to those weighted polarized graphs as in Theorem 2.7 (a),(b). Then we have $\langle \Delta, \Delta \rangle > 0$.*

Remark 2.9. To obtain $\langle \Delta, \Delta \rangle \geq 0$ with our approach, the assumption that $\Phi(\overline{G}_y)(\lambda_y) \geq 0$ for all $y \in B$ with $\overline{G}_y \cong \overline{\mathbf{N}}$ is necessary. Indeed, if λ is a weight of $\mathbf{N}$ such that all the edges have the same length l, then we can calculate it to obtain

$$\Phi(\overline{\mathbf{N}})(\lambda) = -\frac{2}{7} l < 0.$$

Hence the summation of $\Phi(\overline{G}_y)(\lambda_y)$ may be negative. Nevertheless, if we know all the dual graphs for a given $f : \mathcal{X} \to B$, we can concretely check whether it satisfies the assumption by the explicit formula given in Proposition 2.3 (and the contraction lemma).

3. Calculation of the invariant for hyperelliptic polarized graphs

The purpose of this section is to find an explicit formula for the invariant ψ for a certain kind of graphs, called hyperelliptic graphs.

3.1. **Hyperelliptic polarized graphs.** Let us recall the notion of hyperelliptic graph used in [11].[2] It was defined as a pair (G, ι) of a connected graph G and an automorphism on G of order 2 with the following properties:

(a) G is not a one-point graph.
(b) Any edge is a line segment (i.e., there is no self-loop).
(c) $\iota(e) \neq e$ for any $e \in G$.
(d) The quotient graph $G/\langle \iota \rangle$ is a tree.
(e) If a vertex v is not fixed by ι, then the valence b_v is at least 3.

It is natural to ask whether a graph G can have two different automorphisms ι and ι' such that both (G, ι) and (G, ι') are hyperelliptic graphs in the above sense. The following lemma is the answer to it:

Lemma 3.1. *For a graph G, if ι and ι' satisfy the above conditions, namely, (G, ι) and (G, ι') are hyperelliptic graphs, then $\iota = \iota'$.*

Proof. Since a hyperelliptic graph has even number of edges, we can write $\#\operatorname{Ed}(G) = 2n$ $(n \in \mathbb{N})$. We will prove our assertion by induction on n. If $n = 1$, it is trivial. Suppose we have our assertion up to $n = m > 1$, and let G be a graph with $2(m+1)$ edges and with hyperelliptic involutions ι and ι'.

Since (G, ι) is a hyperelliptic graph, there is a vertex $v \in \operatorname{Vert}(G)$ such that its image in the quotient graph $G/\langle \iota \rangle$ is an end. In particular the valence at v is 2. Since (G, ι') is also a hyperelliptic graph on the other hand, we find v is fixed by ι' by (e) above. Therefore if

[2] The notion of hyperelliptic graphs seems to first appear in the author's preprint "Bogomolov's Conjecture for Hyperelliptic Curves over Function Fields, arXiv:math/9903066". The paper [11] is a totally revised version of it. The "hyperelliptic graphs" can be also found in [2, §5], in which the "2-edge-connected hyperelliptic graphs" would be the hyperelliptic graphs in our sense.

e_1 and e_2 are the edges with v as an extremity, then $\iota(e_1) = \iota'(e_1) = e_2$, and in particular $\{e_1, e_2\}$ is stable by both ι and ι'. Accordingly, we can induce the hyperelliptic actions ι and ι' on the contraction $G_{\{e_1,e_2\}}$ of $\{e_1, e_2\}$. By the induction hypothesis, the two induced actions on $G_{\{e_1,e_2\}}$ coincide with each other. Since the action of ι and ι' coincides on the subgraph of G generated by $\{e_1, e_2\}$, we have actually $\iota = \iota'$ on G. □

Thus we can make the following definition:

Definition 3.2. A graph G is called a *hyperelliptic* graph if it is the one-point graph or if it admits an automorphism ι of order 2 such that (G, ι) is a hyperelliptic graph in the sense above. We call ι, which is unique by the above lemma, the *hyperelliptic involution* of G.

In the sequel, *let ι stand for the hyperelliptic involution*. Note that the one-point graph is also a hyperelliptic graph in the definition here, but everything is trivial for it.

Definition 3.3. Let $\overline{G} = (G, q)$ be a polarized graph. We call it a *hyperelliptic polarized graph* if

(a) G is hyperelliptic,
(b) ι preserves the polarization q, and
(c) $q(v) = 0$ for any v with $\iota(v) \neq v$.

Let λ be a weight on G. We sat $(\overline{G}, \lambda)$ is a *hyperelliptic weighted polarized graph* if $\overline{G}$ is a hyperelliptic polarized graph and λ is invariant by ι.

Any graph invariants discussed so far are considered for hyperelliptic polarized graphs of course. As far as we are focusing on hyperelliptic objects, they should be regarded as a function on the set of the ι-invariant weights. To be precise, let $\mathcal{W}(G/\langle\iota\rangle)$ be the linear subspace of $\mathcal{W}(G)$ consisting of ι-invariant elements, and put

$$\mathcal{W}_{>0}(G/\langle\iota\rangle) := \mathcal{W}_{>0}(G) \cap \mathcal{W}(G/\langle\iota\rangle).$$

In the rest of this section, we consider such graph invariants as $\epsilon(\overline{G})$, $\psi(\overline{G})$ and so on as a functions on $\mathcal{W}_{>0}(G/\langle\iota\rangle)$.

Let $\overline{G}$ be a nontrivial hyperelliptic polarized graph. We can introduce the notion of *subtype* of $[e] := \{e, \iota(e)\}$ for any $e \in \mathrm{Ed}(G)$. Let $\overline{G}^{[e]}$ be the contraction of $\mathrm{Ed}(G) \setminus [e]$. It is also a hyperelliptic polarized graph and it has exactly two vertices, say v and w. In particular, we find that e is of type 0. Then we define the *subtype* j of $[e]$ by

$$j := \min\{q'(v), q'(w)\},$$

where q' is the polarization of $\overline{G}^{[e]}$. Note that $0 \leq j \leq [(g-1)/2]$, where g denotes the genus of $\overline{G}$.

Finally, we define a function $\xi_j(\overline{G})$ on $\mathcal{W}_{>0}(G/\langle\iota\rangle)$, for $0 \leq j \leq [(g-1)/2]$, by

$$\xi_j(\overline{G})(\lambda) := \begin{cases} 2 \displaystyle\sum_{[e]\,:\,\text{of subtype } 0} \lambda(e) & \text{if } j = 0, \\ \displaystyle\sum_{[e]\,:\,\text{of subtype } j} \lambda(e) & \text{otherwise.} \end{cases}$$

Note that

$$\xi_0(\overline{G}) + 2\sum_j \xi_j(\overline{G}) = \delta(\overline{G}).$$

Remark 3.4. Even if $(\overline{G}, \lambda)$ is just *equivalent to* a hyperelliptic weighed polarized graph $(\overline{G}', \lambda')$, we can define ξ_j by $\xi_j(\overline{G})(\lambda) := \xi_j(\overline{G}')(\lambda')$.

3.2. **Calculation of ψ for hyperelliptic graphs.** The goal of this subsection is to describe ψ explicitly, namely, to show the following theorem.

Theorem 3.5. *Let $\overline{G}$ be a hyperelliptic polarized graph. Then we have*

$$\psi(\overline{G}) = \frac{g-1}{2g+1}\delta_0(\overline{G}) + \sum_{j=1}^{\left[\frac{g-1}{2}\right]} \frac{6j(g-1-j)}{2g+1}\xi_j(\overline{G})$$

as functions on $\mathcal{W}_{>0}(G/\langle\iota\rangle)$.

Before going on to the proof for general $\overline{G}$, let us check first that it holds for the *minimal* irreducible hyperelliptic polarized graphs. We say that an irreducible hyperelliptic polarized graph $\overline{G}$ is *minimal* if G is nontrivial and if G *with the hyperelliptic involution* is minimal in the sense of [11, Definition 2.13]. Note that the minimality is the notion on the graph only, independent of the polarization. We do not repeat the definition here, but should note that G is minimal if and only if the contraction of any non-empty subset of $\mathrm{Ed}(G)$ cannot be an irreducible hyperelliptic graph.

In order to know what the minimal hyperelliptic graphs are like, let us see what happens when edges are contracted (See [11, §2.1] for details.). Let e be an edge of an irreducible hyperelliptic graph G. For $i = 0, 1, 2$, we say that e is *i-jointed* if e and $\iota(e)$ have exactly i common vertices. Suppose that e is 2-jointed. Then the graph $H := e \cup \iota(e)$ generated by the edges is ι-equivariant subgraph. Since $G/\langle\iota\rangle$ is a tree, we find $H/\langle\iota\rangle$ is a subtree consisting of one edge. Accordingly, the two vertices of H are ι-fixed vertices, and H is an irreducible component of G. By the irreducibility of G, it must coincide with G. In this case, G is a minimal hyperelliptic graph of first Betti number 1 (c.f. Figure 4). If e

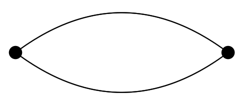

FIGURE 4

is 1-jointed, then we can find that the hyperelliptic graph obtained by contracting $\{e, \iota(e)\}$ must be reducible. If e is 0-jointed then we can see that the contraction of $\{e, \iota(e)\}$ is again an irreducible hyperelliptic graph.

Thus it is true that an irreducible hyperelliptic graph G is minimal if and only if it does not have a 0-jointed edge. Now it is not difficult to classify them and find their concrete configurations. In fact, let n denote the first Betti number of G. If $n = 1$, then it must be the graph as in Figure 4 mentioned above. Suppose $n \geq 2$. In this case, G has 1-jointed edges only, and we can find that G is the graph as in Figure 5. Note for minimal G with

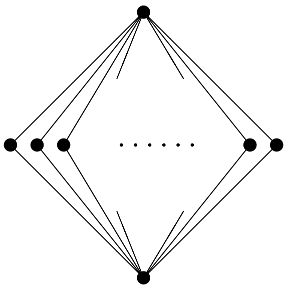

FIGURE 5

$n \geq 2$, we have

$$2(n+1) = \#\,\mathrm{Ed}(G) \tag{3.5.12}$$

although we have only

$$2(n+1) \leq \#\,\mathrm{Ed}(G) \tag{3.5.13}$$

for not necessarily minimal irreducible hyperelliptic graphs.

Now let us consider the invariants for minimal hyperelliptic polarized graphs. In the following examples, let $\overline{G} = (G, q)$ be a hyperelliptic polarized graph of genus g.

Example 3.6. Assume that G has the first Betti number 1 (cf. Figure 4). Let e be an edge, and let j be the subtype of the pair of the edges. For $\lambda \in \mathcal{W}_{>0}(G/\langle\iota\rangle)$, let l denote the length of e. Note $\delta(\overline{G})(\lambda) = 2l$. Then by [11, Theorem 3.14], we have

$$\epsilon(\overline{G})(\lambda) = \frac{2(g-1) + 6j(g-1-j)}{3g} l$$

and

$$r(\overline{G})(\lambda) = 4j(g-1-j)l,$$

and hence the equality in Theorem 3.5 follows from the definition of ψ.

Example 3.7. Let G be the minimal hyperelliptic graph of first Betti number $n \geq 2$ (cf. Figure 5). Let $v, \iota(v)$ be the vertices not fixed by ι. Let $e_0, \ldots, e_n$ be the edges with v as an extremity, and let w_i be the other extremity of e_i for $i = 0, \ldots, n$. Let q be a polarization such that the polarized graph $\overline{G} := (G, q)$ is a polarized hyperelliptic graph (c.f. Definition 3.3). Let g be the genus of $\overline{G} := (G, q)$. Put $q_i := q(w_i)$. Note that $q(v) = q(\iota(v)) = 0$ by the definition of the hyperelliptic polarized graph. Then the canonical divisor K is given by

$$K = \sum_{i=0}^{n} 2q_i w_i + (n-1)(v + \iota(v)).$$

Note $\sum_{i=0}^{n} q_i + (n-1) = g - 1$.

Let $\lambda \in \mathcal{W}_{>0}(G/\langle\iota\rangle)$ and put $l_i = \lambda(e_i)$. Let σ_k denote the k-th elementary symmetric polynomial on $l_0, \ldots, l_n$. Then we can find

$$\epsilon(\overline{G})(\lambda) = \frac{g-1}{3g}\delta_0(\overline{G})(\lambda) + \frac{2(g-1)(n-1)}{3g}\frac{\sigma_{n+1}}{\sigma_n} + \frac{2}{g}\sum_{j=1}^{\left[\frac{g-1}{2}\right]} j(g-1-j)\xi_j(\overline{G})(\lambda)$$

by [11, Theorem 3.14] if we take account of the following:

- $\deg(K) = 2g-2$,
- $\mathrm{tp}([e_i])(\deg(K) - \mathrm{tp}([e_i])) = 4q_i(g-1-q_i)$, where tp is the one defined just before [11, Theorem 3.14], and
- [11, Example 3.12].

Further we can easily see

$$\begin{aligned} r_G(v,\iota(v))(\lambda) &= \frac{2\sigma_{n+1}}{\sigma_n}, \\ r_G(w_i,v)(\lambda) = r_G(w_i,\iota(v))(\lambda) &= \frac{l_i}{2} + \frac{\sigma_{n+1}}{\sigma_n}, \\ r_G(w_i,w_j)(\lambda) &= \frac{1}{2}(l_i + l_j) \quad (i \neq j) \end{aligned}$$

and hence we have

$$\begin{aligned} r_G(\overline{G})(\lambda) &= 4\sum_{i=0}^{n} q_i(g-1-q_i)l_i + 4(n-1)(g-1)\frac{\sigma_{n+1}}{\sigma_n} \\ &= 4\sum_{j=1}^{\left[\frac{g-1}{2}\right]} j(g-1-j)\xi_j(\overline{G})(\lambda) + 4(n-1)(g-1)\frac{\sigma_{n+1}}{\sigma_n}. \end{aligned}$$

According to them, we can directly obtain from the definition of $\psi(\overline{G})$ the equality

$$\psi(\overline{G}) = \frac{g-1}{2(2g+1)}\delta_0(\overline{G}) + \frac{6}{2g+1}\sum_{j=1}^{\left[\frac{g-1}{2}\right]} j(g-1-j)\xi_j(\overline{G})$$

as expected.

Now we put

$$\Psi(\overline{G}) := \frac{g-1}{2g+1}\delta_0(\overline{G}) + \sum_{j=1}^{\left[\frac{g-1}{2}\right]} \frac{6j(g-1-j)}{2g+1}\xi_j(\overline{G}) - \psi(\overline{G}).$$

Our goal is $\Psi(\overline{G}) = 0$.

Let L_G be the polynomial function on $\mathcal{W}_{>0}(G/\langle\iota\rangle)$ defined at the beginning of [11, §3.1]. Here we mean by polynomial function in the following sense: By the definition, $\mathcal{W}(G/\langle\iota\rangle)$ is the dual vector space of the vector space with the basis $\mathrm{Ed}(G)/\langle\iota\rangle$, and the polynomial here means that on the dual basis of $\mathrm{Ed}(G)/\langle\iota\rangle$.

Remark 3.8. Let n denote the first Betti number of G. Then the following functions on $\mathcal{W}_{>0}(G/\langle\iota\rangle)$ are homogeneous polynomial functions of deg $\leq n+1$:

(1) $L_G g_{\overline{G}}(o, v)$ if o is an ι-invariant vertex (cf. [11, Lemma 3.9]),
(2) $L_G r_G(o, v)$ if o is an ι-invariant vertex (cf. [11, Lemma 3.10]),
(3) $L_G \epsilon(\overline{G})$ (cf. [11, Proposition 3.11]).

The following lemma tells us the same thing holds for $\Psi(\overline{G})$:

Lemma 3.9. *Let n denote the first Betti number of G. Then as a function on $\mathcal{W}_{>0}(G/\langle\iota\rangle)$, $L_G\Psi(\overline{G})$ is a homogeneous polynomial function of* $\deg \leq n+1$.

Proof. It is enough to show $L_G\psi(\overline{G})$ is such a polynomial. Taking account of Remark 3.8 (3), we are reduced to show that $L_G r(\overline{G})$ is a homogeneous polynomial function of $\deg \leq n+1$.

Step 1. Let v be a vertex. Take an ι-fixed vertex o. We know

$$g_{\overline{G}}(v, v) = r_G(o, v) - g_{\overline{G}}(o, o) + 2g_{\overline{G}}(o, v).$$

By Remark 3.8 (1) and (2), we see that $L_G g_{\overline{G}}(v, v)$ is a homogeneous polynomial function of $\deg \leq n+1$.

Step 2. Let o be an ι-fixed vertex. Then, we see that

$$L_G c(\overline{G}) = L_G g_{\overline{G}}(K, o) + L_G g_{\overline{G}}(o, o)$$

and $L_G\epsilon(\overline{G})$ are homogeneous polynomial functions of $\deg \leq n+1$ by Remark 3.8 (1) and (3). Since

$$\epsilon(\overline{G}) = 2\deg(K)c(\overline{G}) - g_{\overline{G}}(K, K),$$

we find $L_G g_{\overline{G}}(K, K)$ is a homogeneous polynomial function of $\deg \leq n+1$.

Step 3. Using the equality

$$r_G(v, w) = g_{\overline{G}}(v, v) - 2g_{\overline{G}}(v, w) + g_{\overline{G}}(w, w)$$

of [12, (3.5.1)], we obtain

$$r(\overline{G}) = r_G(K, K) = \sum_{v,w} d_v d_w (g_{\overline{G}}(v, v) + g_{\overline{G}}(w, w)) - 2g_{\overline{G}}(K, K).$$

Then by Step 1 and Step 2, we find that $L_G r(\overline{G})$ is a homogeneous polynomial function of $\deg \leq n+1$. □

Now we are ready to prove $\Psi(\overline{G}) = 0$ for all hyperelliptic $\overline{G}$. The proof is essentially same as that of [11, Theorem 3.14], and we will give only a sketch.

We will show it by induction on $m := \#\mathrm{Ed}(G/\langle\iota\rangle)$. Since we know the sum formula, we may assume G is irreducible. If $m \leq 3$, then G is minimal and our assertion has already been obtained in Example 3.6 and 3.7, hence we may assume $m \geq 4$.

Let n denote the first Betti number of G and let d denote the degree of the homogeneous polynomial function $F := L_G\Psi(\overline{G})$. We have $d \leq n+1 \leq m$ by Lemma 3.9 and (3.5.13).

If $d = m$, then $n+1 = m$ and hence G is a minimal hyperelliptic graph of first Betti number $d-1$ by (3.5.12). In this case, we are done in Example 3.7.

Suppose $d < m$. Now we note the following claim (cf. [11, Claim 1 in §3.4]).

Claim 3.9.14. Let P be a homogeneous polynomial on $Y_1, \ldots, Y_m$ of degree d with $d < m$. Suppose for each i that $P(a_1, \ldots, a_m) = 0$ if $a_i = 0$. Then, we have $P = 0$ as a polynomial.

By the contraction lemma and induction hypothesis, F satisfies the condition of the above claim. Therefore we have $F = 0$, and thus we complete the proof of Theorem 3.5.

4. Graphically hyperelliptic curves, a conjecture and a result

In this section, we give some applications of the results in the previous sections. We repeat that K is a function field of a smooth projective curve B over an algebraically closed field k. We assume that X is a smooth projective curve over K of genus $g \geq 2$ with a semistable model $f : \mathcal{X} \to B$. Let $(\overline{G}_y, \lambda_y)$ be the weighted polarized dual graph over y and let $(\overline{G}_{y0}, \lambda_{y0})$ be the contraction of the edges of positive type.

Definition 4.1. We call X or f a *graphically hyperelliptic* curve if $(\overline{G}_{y0}, \lambda_{y0})$ for any y is equivalent to a hyperelliptic weighted polarized graph.

A hyperelliptic curve is a graphically hyperelliptic curve (cf. [11, §4.3]). For a graphically hyperelliptic curve X, we define

$$\xi_j(X_y) := \xi_j(\overline{G}_{y0})(\lambda_{y0}),$$

which is well-defined (cf. Remark 3.4). Note that if f is the relatively minimal model for a hyperelliptic curve with the hyperelliptic involution ι, then $\xi_j(X_y)$ is nothing but the quantity $\xi_j(\mathcal{X}_y)$ in [3] or [10]. We say the *Cornalba-Harris equality holds for X* if

$$(8g+4)\deg\left(f_*\omega_{\mathcal{X}/B}\right) = g\xi_0(X) + \sum_{j=1}^{\left[\frac{g-1}{2}\right]} 2(j+1)(g-j)\xi_j(X) + \sum_{i=1}^{\left[\frac{g}{2}\right]} 4i(g-i)\delta_i(X)$$

holds, where $\xi_j(X) = \sum_y \xi_j(X_y)$ and $\delta_i(X) = \sum_y \delta_i(X_y)$. Note also that $\deg\left(f_*\omega_{\mathcal{X}/B}\right)$ depends only on X. A hyperelliptic curve is graphically hyperelliptic and the Cornalba-Harris equality holds by [3] in characteristic 0, and by [10] in positive characteristic.

For the height of the canonical Gross-Schoen cycle of a graphically hyperelliptic curve, we have the following assertion.

Theorem 4.2 ($\mathrm{char}(k) \geq 0$)**.** *Let X be a graphically hyperelliptic curve. Then the inequality*

$$(8g+4)\deg\left(f_*\omega_{\mathcal{X}/B}\right) \geq g\xi_0(X) + \sum_{j=1}^{\left[\frac{g-1}{2}\right]} 2(j+1)(g-j)\xi_j(X) + \sum_{i=1}^{\left[\frac{g}{2}\right]} 4i(g-i)\delta_i(X). \tag{4.2.15}$$

hold if and only if $\langle\Delta,\Delta\rangle \geq 0$. Moreover, the equality, namely the Cornalba-Harris equality holds for X, if and only if $\langle\Delta,\Delta\rangle = 0$.

Proof. By (1.2.6), the positivity of $\langle\Delta,\Delta\rangle$ is equivalent to that of

$$\left(\omega_{\mathcal{X}/B}\cdot\omega_{\mathcal{X}/B}\right) - \sum_y \psi(\overline{G}_y)(\lambda_y). \tag{4.2.16}$$

Let $\overline{G}_{y+}$ be the contraction of all the edges of type 0 of $\overline{G}_y$. Then by the sum formula we have

$$\psi(\overline{G}_y)(\lambda_y) = \psi(\overline{G}_{y0})(\lambda_{y0}) + \psi(\overline{G}_{y+})(\lambda_{y+}),$$

where λ_{y0} and λ_{y+} are the induced weights on G_{y0} and G_{y+} by the contractions respectively. Since G_{y+} is a tree, we have, by (1.2.5),

$$\psi(\overline{G}_{y+}) = \sum_{i=1}^{[\frac{g}{2}]} \left(\frac{12i(g-i)}{2g+1} - 1 \right) \delta_i(\overline{G}_{y+}).$$

On the other hand, since $\overline{G}_{y0}$ is a hyperelliptic polarized graph, we have, by Theorem 3.5,

$$\psi(\overline{G}_{y0}) = \frac{g-1}{2g+1}\delta_0(\overline{G}_{y0}) + \sum_{j=1}^{[\frac{g-1}{2}]} \frac{6j(g-1-j)}{2g+1}\xi_j(\overline{G}_{y0}).$$

Accordingly we find that (4.2.16) is equal to

(4.2.18)

$$\left(\omega_{\mathcal{X}/B} \cdot \omega_{\mathcal{X}/B}\right) - \frac{g-1}{2g+1}\delta_0(X) + \sum_{j=1}^{[\frac{g-1}{2}]} \frac{6j(g-1-j)}{2g+1}\xi_j(X) + \sum_{i=1}^{[\frac{g}{2}]} \left(\frac{12i(g-i)}{2g+1} - 1 \right) \delta_i(X).$$

Now using Noether's formula (cf. [7, §1 (1)]), we see that (4.2.15) is nothing but the non-negativity of (4.2.18), and the equality condition also follows immediately. □

Thus we have given an alternative proof of the following known assertion, by the direct calculation of the height.

Corollary 4.3 (char$(k) \geq 0$)**.** *If f is hyperelliptic, then* $\langle \Delta, \Delta \rangle = 0$.

Remark 4.4. Under the assumption of the characteristic 0, we have (4.2.15) for a graphically hyperelliptic curve f by [9, Corollary 3.3]. It is conjectured that $\langle \Delta, \Delta \rangle \geq 0$ in positive characteristic, and hence inequality (4.2.15) should hold for a graphically hyperelliptic curve in all characteristics.

Thus if X is hyperelliptic, then it is graphically hyperelliptic, and the Cornalba-Harris equality holds for it, namely $\langle \Delta, \Delta \rangle = 0$. It is natural to ask how about the converse:

Conjecture 4.5 (char$(k) \geq 0$)**.** A graphically hyperelliptic curve with $\langle \Delta, \Delta \rangle = 0$ should be a hyperelliptic curve.

Although we do not know whether it holds true or not in general, we can show it to be true when the genus is 3.

Theorem 4.6 (char$(k) \geq 0$)**.** *If $g = 3$, the above conjecture is true.*

Proof. By our assumption, any polarized dual graph, after contraction of all the edges of positive type, is equivalent to a hyperelliptic polarized graph. A hyperelliptic graph cannot be of configuration like $\mathbf{N}$, $\mathbf{E}_1$ nor $\mathbf{E}_2$. Therefore by the Corollary 2.8, we see that if it is not hyperelliptic, then $\langle \Delta, \Delta \rangle > 0$, which contradicts Theorem 4.2. Thus we have our assertion. □

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